

ALGEBRA IN A GROTHENDIECK TOPOS: INJECTIVITY IN QUASI-EQUATIONAL CLASSES

M. Mehdi EBRAHIMI

*Department of Mathematical Sciences, McMaster University, Hamilton, Ontario, Canada**

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This paper gives a study of injectivity and some related notions in quasi-equational classes of algebras in an arbitrary Grothendieck topos \mathbf{E} . The main purpose is to describe the relationship between the class $\text{mod } \Sigma$ of models of a set Σ of quasi-equations in the category of sets \mathbf{Ens} and the corresponding class $\text{mod}(\Sigma, \mathbf{E})$ of models of Σ in \mathbf{E} with respect to residual smallness, boundedness of essential extensions and injectivity. The basic nature of our results is that, for any given Σ , whatever holds in \mathbf{Ens} , concerning these notions, also holds in \mathbf{E} . In particular, this substantially improves the earlier results of Howlett [5] regarding the existence of enough injectives in $\text{mod}(\Sigma, \mathbf{E})$.

1. Preliminaries

1.1. Algebras in a category. Let \mathbf{E} be a finitely complete category (in particular, it has a terminal object 1). Given a family $\tau = (n_\lambda)_{\lambda \in \Omega}$ of finite cardinal numbers n_λ , indexed by a set Ω , recall that an algebra in \mathbf{E} is an entity $A = (|A|, (\lambda_A)_{\lambda \in \Omega})$, where $|A|$ is an object of \mathbf{E} , called the underlying object of the algebra A and, for each $\lambda \in \Omega$, the λ -th operation $\lambda_A: |A|^{n_\lambda} \rightarrow |A|$ of A is a morphism in \mathbf{E} , n_λ being the arity of λ_A ; the family $\tau = (n_\lambda)_{\lambda \in \Omega}$ is called the type of A . A homomorphism $h: A \rightarrow B$ from an algebra $A = (|A|, (\lambda_A)_{\lambda \in \Omega})$ to an algebra $B = (|B|, (\lambda_B)_{\lambda \in \Omega})$ is a morphism $|h|: |A| \rightarrow |B|$ such that the following diagram commutes, for each $\lambda \in \Omega$:

$$\begin{array}{ccc} |A|^{n_\lambda} & \xrightarrow{|h|^{n_\lambda}} & |B|^{n_\lambda} \\ \lambda_A \downarrow & & \downarrow \lambda_B \\ |A| & \xrightarrow{|h|} & |B| \end{array}$$

* Presently at the Department of Mathematics, National University of Iran, Eveen, Tehran, Iran.

The collection of all algebras (of the type τ) in \mathbf{E} and homomorphisms between them forms a category denoted by $\text{Alg}(\tau)\mathbf{E}$ (or by $\text{Alg}(\tau)$ if $\mathbf{E} = \mathbf{Ens}$).

For $A \in \text{Alg}(\tau)\mathbf{E}$ and any natural number n , $\mathbf{E}(|A|^n, |A|)$ can easily be made into an algebra of the type τ in \mathbf{Ens} , by defining the λ -th operation as

$$\lambda(\phi_1, \dots, \phi_{n_\lambda}) = \lambda_A \prod_{i=1}^{n_\lambda} \phi_i$$

for any $\phi_i: |A|^n \rightarrow |A|$ ($i = 1, \dots, n_\lambda$), where $\prod_{i=1}^{n_\lambda} \phi_i$ is the morphism $|A|^n \rightarrow |A|^{n_\lambda}$ determined by the ϕ_i . Let F be the absolutely free algebra of the type τ on a set $X = \{x_1, \dots, x_n\}$ of n elements. Extend the map $x_i \rightsquigarrow \text{pr}_i$ ($\text{pr}_i: |A|^n \rightarrow |A|$ the projections) from X to $\mathbf{E}(|A|^n, |A|)$ freely to a homomorphism $\phi: F \rightarrow \mathbf{E}(|A|^n, |A|)$ and denote $\phi(P)$ by P_A for any $P \in F$. For a quasi-equation

$$\sigma := \bigwedge_{i=1}^k (p_i = q_i) \rightarrow (p = q),$$

we say that A satisfies σ , written as $A \models \sigma$, iff the pullback

$$\begin{array}{ccccc} & & \text{Eq}_1(p_{1A}, q_{1A}) & & \\ & \nearrow & \vdots & \searrow & \\ P & \xrightarrow{\quad} & \text{Eq}_i(p_{iA}, q_{iA}) & \xrightarrow{\quad} & |A|^n \xrightarrow[p_{iA}]{q_{iA}} |A| \\ & \searrow & \vdots & \nearrow & \\ & & \text{Eq}_k(p_{kA}, q_{kA}) & & \end{array}$$

of the equalizers

$$\text{Eq}_i(p_{iA}, q_{iA}) \rightarrow |A|^n \xrightarrow[p_{iA}]{q_{iA}} |A| \quad (i = 1, \dots, k)$$

factors through the equalizer

$$\text{Eq}(p_A, q_A) \rightarrow |A|^n \xrightarrow[p_A]{q_A} |A|.$$

In particular, A satisfies an equation $(p = q)$ iff $p_A = q_A$. The full subcategory of $\text{Alg}(\tau)\mathbf{E}$ given by the class of all algebras in \mathbf{E} satisfying Σ will be denoted by $\text{mod}(\Sigma, \mathbf{E})$ (or $\text{mod } \Sigma$ if $\mathbf{E} = \mathbf{Ens}$) and is called a quasi-equational class (or an equational class if Σ is a set of equations). We note that, if \mathbf{E} is a topos, then the above definition of $A \models \sigma$ coincides with the usual notion of satisfaction as defined for an arbitrary first order sentence in a topos. From now on Σ always denotes a set of quasi-equations.

Let $k: \mathbf{E} \rightarrow \mathbf{F}$ be a functor, preserving finite limits; then k induces another functor $f: \mathbf{A} \rightarrow \mathbf{B}$ in $\text{Alg}(\tau)\mathbf{E}$. Since k preserves finite limits, it preserves pullback and equalizer diagrams; and hence if σ is a quasi-equation and $A \models \sigma$, for $A \in \text{Alg}(\tau)\mathbf{E}$, then $kA \models \sigma$. We thus get a functor

$$k|_{\text{mod}(\Sigma, \mathbf{E})}: \text{mod}(\Sigma, \mathbf{E}) \rightarrow \text{mod}(\Sigma, \mathbf{F})$$

for any given set Σ of quasi-equations.

1.2. Lemma. Let \mathbf{E} have a set Φ of generators. Then, for any $A \in \text{Alg}(\tau)\mathbf{E}$ and any set Σ of quasi-equations, $A \in \text{mod}(\Sigma, \mathbf{E})$ iff $\bar{h}_G(A) \in \text{mod } \Sigma$ for each $G \in \Phi$, where $\bar{h}_G = \mathbf{E}(G, -)$.

Proof. That $A \in \text{mod}(\Sigma, \mathbf{E})$ implies $\bar{h}_G(A) \in \text{mod } \Sigma$ is clear, by what has been discussed above. Conversely, let

$$\sigma := \bigwedge_{i=1}^k (p_i = q_i) \rightarrow (p = q)$$

be any quasi-equation and $A \in \text{Alg}(\tau)\mathbf{E}$ such that $\bar{h}_G(A) \models \sigma$ for all $G \in \Phi$. Consider the diagram

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow j & & \\ E & \xrightarrow{i} & |A|^n & \xrightleftharpoons[p_A]{q_A} & |A| \\ & & \downarrow p_A \quad \downarrow q_{iA} & & \\ & & |A| & & \end{array}$$

with P the pullback of the equalizers of the pairs (p_{iA}, q_{iA}) . By the hypothesis on A , $\bar{h}_G(p_{iA}j) = \bar{h}_G(q_{iA}j)$ for all $G \in \Phi$ which this implies that $p_{iA}j = q_{iA}j$, the latter because the set Φ of generators is collectively faithful, and hence j factors through i , by definition of equalizers; thus $A \models \sigma$.

1.3. Algebras in a Grothendieck topos. Let \mathbf{C} be a small category. The category $\hat{\mathbf{C}} = \text{Ens}^{\mathbf{C}^*}$ (\mathbf{C}^* the dual category of \mathbf{C}) is called the category of presheaves on \mathbf{C} . Recall that, up to equivalence, a category \mathbf{E} is a *Grothendieck topos* iff it is a full subcategory of some $\hat{\mathbf{C}}$ for which there is a reflection functor $R: \hat{\mathbf{C}} \rightarrow \mathbf{E}$ (i.e. a left adjoint to the inclusion function $\mathbf{E} \xrightarrow{i} \hat{\mathbf{C}}$) which is also left exact (i.e. preserves finite limits) (Johnstone [6], p. 12 and p. 105). Let

$$\mathbf{E} \begin{array}{c} \xrightarrow{i} \\ \xleftarrow{R} \end{array} \hat{\mathbf{C}}$$

be a Grothendieck topos. The discussion in the last sections, in particular, shows that the category $\text{Alg}(\tau)\hat{\mathbf{C}}$ is isomorphic to the category of all $\text{Alg}(\tau)$ -valued presheaves on \mathbf{C} ; and since R preserves finite limits, it can be lifted to $\bar{R}: \text{mod}(\Sigma, \hat{\mathbf{C}}) \rightarrow \text{mod}(\Sigma, \mathbf{E})$ (we denote \bar{R} by the same letter R). Moreover, since \mathbf{E} has a set of generators, namely $\{R(h_U): U \in \mathbf{C}\}$, where $h_U = (-, U)$ is the presheaf represented by U , Lemma 1.3 implies that $A \in \text{mod}(\Sigma, \mathbf{E})$ iff $AU \in \text{mod } \Sigma$ for all $U \in \mathbf{C}$.

A natural question to ask would be, what is the relationship between the

behaviour of a certain classical algebraic notion in $\text{mod } \Sigma$ and in $\text{mod}(\Sigma, \mathbf{E})$. In this paper, we primarily consider the notion of injectivity, which has been extensively investigated for the case of equational classes of algebras in Ens (e.g. [1], [7]), and show that the properties of $\text{mod } \Sigma$, regarding this notion, survive the passage to $\text{mod}(\Sigma, \mathbf{E})$, for a set Σ of quasi-equations and an *arbitrary* Grothendieck topos \mathbf{E} . For example, we prove that $\text{mod}(\Sigma, \mathbf{E})$ has enough injectives iff it is residually small and pushouts transfer monomorphisms, which is a counterpart of a result for equational classes of algebras in Ens ; further, we show that $\text{mod}(\Sigma, \mathbf{E})$ has enough injectives iff $\text{mod } \Sigma$ has enough injectives.

2. The adjointness of $\hat{\mathbf{C}}$

2.1. Here, we construct a pair of adjoint functors

$$\hat{\mathbf{C}} \begin{array}{c} \xrightarrow{G} \\ \xleftarrow{H} \end{array} \text{Ens}^{|\mathbf{C}|}$$

with G a left adjoint to H , where $|\mathbf{C}|$ denotes the set of object of \mathbf{C} .

Define G by $GP = (PU)_{U \in \mathbf{C}}$, for $P \in \hat{\mathbf{C}}$, and for any map $f: P \rightarrow Q$ in $\hat{\mathbf{C}}$, $Gf = (f_U)_{U \in \mathbf{C}}$. That G is a functor is easily checked.

Define H by: for any $B = (B_V)_{V \in \mathbf{C}}$ in $\text{Ens}^{|\mathbf{C}|}$, $HB = \bar{B}$ is the presheaf given as follows:

$$BU = \prod_{V \in \mathbf{C}} B_V^{(V, U)}$$

for each $V \in \mathbf{C}$ and to define $\bar{B}s: \bar{B}W \rightarrow \bar{B}U$, for each $s: U \rightarrow W$, notice that s induces a natural transformation $s^*: h_U \rightarrow h_W$ with components s_V^* given by $s_V^*(t) = st$ ($V \in \mathbf{C}$), for $t: V \rightarrow U$; then s^* induces a map $s_V^B: B_V^{(V, W)} \rightarrow B_V^{(V, U)}$ (composition from the left). Hence, define $\bar{B}s = \prod_{V \in \mathbf{C}} s_V^B$. One easily checks that \bar{B} is a presheaf. To complete the definition of the functor H , let $f = (f_V)_{V \in \mathbf{C}}$ be any map from $B = (B_V)_{V \in \mathbf{C}}$ to $C = (C_V)_{V \in \mathbf{C}}$. For any pair V and U in \mathbf{C} , define $f_V^U: B_V^{(V, U)} \rightarrow C_V^{(V, U)}$ by $f_V^U(\alpha) = f_V \alpha$ for any $\alpha: (V, U) \rightarrow B_V$. Define $\bar{f} = Hf$ by $\bar{f}_U = \prod_V f_V^U$, ($U \in \mathbf{C}$). That \bar{f} is a natural transformation and H is actually a functor is easily checked.

Next, we define two natural transformations $\eta: 1 \rightarrow HG$ and $\varepsilon: GH \rightarrow 1$. To define η , let $P \in \hat{\mathbf{C}}$; combining the maps $PU \rightarrow PV^{(V, U)}$ given by $a \rightsquigarrow a_V$ with $a_V(s) = (Ps)a$, for $a \in PU$ and $s: V \rightarrow U$, we get a map

$$\eta_{PU}: PU \rightarrow \prod_{V \in \mathbf{C}} PV^{(V, U)} = (HGP) \quad (U \in \mathbf{C}).$$

Define $\eta_P: P \rightarrow HGP$ by $(\eta_P)_U = \eta_{PU}$. It is not hard to check that each η_P and then η are natural transformations. Note that, for $V = U$ and $s = 1_U$, $\bar{a}(s) = a$ and hence η_P is a monomorphism for each $P \in \hat{\mathbf{C}}$. We define ε such that, for each $B = (B_V)_{V \in \mathbf{C}}$, ε_B has the composite

$$e_{pr_u}: \prod_V B_V^{(V, U)} \rightarrow B_U^{(U, U)} \rightarrow B_U$$

as its U -th component ($U \in \mathbf{C}$) and $e(\alpha) = \alpha(1_U)$ for $\alpha : (U, U) \rightarrow B_U$.

2.2. Proposition. *G is a left adjoint to H , with η and ε as front and back adjunctions.*

Proof. It only remains to show that

$$1_G = G \xrightarrow{G \cdot \eta} GHG \xrightarrow{\varepsilon \cdot G} G$$

and

$$1_H = H \xrightarrow{\eta \cdot H} HGH \xrightarrow{H \cdot \varepsilon} H,$$

that is to show, for any $A \in \hat{\mathbf{C}}$ and $B = (B_V)_{V \in \mathbf{C}}$, $\varepsilon_{GA} G \eta_A = 1_{GA}$ and $H \varepsilon_B \eta_{HB} = 1_{HB}$. Let $U \in \mathbf{C}$ and $a \in AU$, we have $(GA)U \rightarrow (GHGA)U \rightarrow (GA)U$ given by $a \rightsquigarrow (\tilde{a}_V)_{V \in \mathbf{C}} \rightsquigarrow e(\tilde{a}_U)$, but $e(\tilde{a}_U) = \tilde{a}_U(1_U) = A(1_U)(a) = a$, and hence $\varepsilon_{GA} G \eta_A = 1_{GA}$. Similarly, one can show $H \varepsilon_B \eta_{HB} = 1_{HB}$.

2.3. Remark. By the definition of G and since H is a right adjoint, they both preserve finite limits, and hence can be lifted to

$$\text{mod}(\Sigma, \hat{\mathbf{C}}) \xrightleftharpoons[H]{G} \text{mod } \Sigma^{|\mathbf{C}|}$$

with \tilde{G} a left adjoint to \tilde{H} .

2.4. Remark. The particular case of the above construction for a monoid \mathbf{M} as the category \mathbf{C} , and hence the category of \mathbf{M} -sets as $\hat{\mathbf{C}}$, is due to Berthiaume [4].

3. Residual smallness

3.1. Definition. A monomorphism $h : A \rightarrow B$ in a category \mathbf{K} is called *essential* iff, for any $g : B \rightarrow C$ in \mathbf{K} , whenever gh is a monomorphism, then so is g .

Let \mathbf{E} be a Grothendieck topos and Σ a set of quasi-equations. Note that, as in the case of \mathbf{Ens} , $\text{mod}(\Sigma, \mathbf{E})$ is closed under direct limits in $\text{Alg}(\tau)\mathbf{E}$ because any colimit in \mathbf{E} is formed by first forming it in $\hat{\mathbf{C}}$ and then reflecting it to \mathbf{E} where the latter preserves the conditions for $A \models \Sigma$.

3.2. Lemma. *In $\text{mod}(\Sigma, \mathbf{E})$,*

(i) *any composite of essential monomorphisms is an essential monomorphism,*
and

(ii) *any direct limit of essential monomorphisms is an essential monomorphism.*

Proof. (i) is trivial. To prove (ii), let $f : A \rightarrow \varinjlim B_\alpha$ be a direct limit in $\text{mod}(\Sigma, \mathbf{E})$ of

essential monomorphisms $f_\alpha: A \rightarrow B_\alpha$ and diagram maps $g_{\alpha\beta}: B_\alpha \rightarrow B_\beta$ ($\beta \geq \alpha$). Since each f_α is a monomorphism, f is a monomorphism, because this is true in \mathbf{Ens} and hence in $\hat{\mathbf{C}}$, and the reflection to \mathbf{E} preserves monomorphisms by exactness. To show that f is essential, let $hf: A \rightarrow D$ be a monomorphism, with D in $\text{mod}(\Sigma, \mathbf{E})$. Then essentialness of f_α implies that all hg_α are monomorphisms, and hence h is a monomorphism; the latter again because of the way direct limits are formed in $\text{mod}(\Sigma, \mathbf{E})$. This proves that f is an essential monomorphism.

3.3. The following is a counterpart of (E3) in [1].

Lemma. *In $\text{mod}(\Sigma, \mathbf{E})$, for any monomorphism $h: A \rightarrow B$ there exists a homomorphism $g: B \rightarrow C$ with gh an essential monomorphism.*

Proof. Take all the congruences Θ on B such that $B/\Theta \in \text{mod}(\Sigma, \mathbf{E})$ and $A \xrightarrow{vh} B/\Theta$ ($v: B \rightarrow B/\Theta$ the quotient map) is a monomorphism. Then by the observation in 3.1 and the exactness argument in the proof of 3.2, any join of a chain of such congruences is again such a congruence; and hence, there exists a maximal such congruence, say Θ_0 . Maximality of Θ_0 then implies that $A \rightarrow B/\Theta_0$ is essential.

3.4. Corollary. *In $\text{mod}(\Sigma, \mathbf{E})$, an algebra A is an absolute retract iff it has no proper essential extension.*

Proof. (\Rightarrow) If $f: A \rightarrow B$ is an essential monomorphism and $h: B \rightarrow A$ is a retraction, then, by essentialness of f , h is a monomorphism, and hence $A \cong B$.

(\Leftarrow) Given any monomorphism $f: A \rightarrow B$, continue it to an essential monomorphism $A \xrightarrow{gf} C$, by the last lemma. By hypothesis on A , gf is an isomorphism and then $(gf)^{-1}g$ is the desired retraction.

3.5. Definition. A category \mathbf{K} is called *residually small* iff, it has a set of cogenerators.

3.6. Definition. A category \mathbf{K} is called *essentially bounded* iff, for each $A \in \mathbf{K}$ there exists, up to isomorphism, only a set of essential extensions in \mathbf{K} .

3.7. Lemma. *For any well powered category with products and a set Φ of generators, residual smallness implies essential boundedness.*

Proof. Let $h: A \rightarrow B$ be any essential monomorphism, and embed $B \xrightarrow{e} \prod_{\alpha \in I} C_\alpha$, for C_α from a suitable set of cogenerators. Then, for any $G \in \Phi$ and a pair of distinct maps $G \xrightarrow{s} A$, we have $ehs \neq eht$, and hence $p_\alpha ehs \neq p_\alpha eht$, for some projection $p_\alpha: \prod C_\alpha \rightarrow C_\alpha$. Pick α_{st} as one such, then $A \rightarrow \prod_{\beta \in J} C_\beta$ is a monomorphism, where $J = \{\alpha_{st}: s \neq t: G \rightarrow A\}$ and $\text{Card } J \leq \text{Card } \bigcup_{G \in \Phi} (G, A)^2$. Essentialness of h implies that $B \rightarrow \prod_J C_\beta$ is a monomorphism, and since there exists only a set of products $\prod_J C_\beta$, we are done.

3.8. Lemma. *For $\text{mod}(\Sigma, \mathbf{E})$, essential boundedness implies residual smallness.*

Proof. For any $A \in \text{mod}(\Sigma, \mathbf{E})$, take all $B_\alpha \leq A$, generated by

$$\alpha: U \amalg U \rightarrow |A| \quad (U \in \mathbf{C})$$

and then continue them to essential extensions

$$B_\alpha \xrightarrow{i_\alpha} A \xrightarrow{f_\alpha} C_\alpha,$$

by Lemma 3.3. The homomorphism $\prod f_\alpha: A \rightarrow \prod C_\alpha$ is a monomorphism, for otherwise there exists some α with $B_\alpha \rightarrow A \rightarrow \prod C_\alpha$ not a monomorphism which is a contradiction to the fact that all $B_\alpha \rightarrow C_\alpha$ are monomorphisms. Since there exists, up to isomorphisms, only a set of B generated by some $U \amalg U \rightarrow |B|$, and only a set of essential extensions of those B , any set representing, up to isomorphism, all essential extensions of such B is a cogenerating set; hence, we are done.

3.9. Corollary. *For $\text{mod}(\Sigma, \mathbf{E})$, essential boundedness is equivalent to residual smallness.*

Proof. One way this is true by the last lemma, and since $\text{mod}(\Sigma, \mathbf{E})$ has a set of generators, namely the $\text{mod}(\Sigma, \mathbf{E})$ -free algebras on the reflection of representable presheaves h_U ($U \in \mathbf{C}$), Lemma 3.7 implies the converse.

3.10. Proposition. *$\text{mod}(\Sigma, \mathbf{E})$ is residually small iff $\text{mod } \Sigma$ is residually small.*

Proof. (\Rightarrow) Consider the following pair of adjoint functors:

$$\text{mod}(\Sigma, \mathbf{E}) \xrightleftharpoons[\Delta]{\Gamma} \text{mod } \Sigma$$

where $\Gamma = (1, -)$ and Δ left exact, left adjoint to Γ ; in fact Δ is the composite

$$\text{mod } \Sigma \xrightarrow{\Delta_0} \text{mod } \Sigma^{|C|} \xrightarrow{|\bar{H}|} \text{mod}(\Sigma, \hat{\mathbf{C}}) \xrightarrow{R} \text{mod}(\Sigma, \mathbf{E})$$

where Δ_0 takes $\text{mod } \Sigma$ to the constant families induced by $|C|$ and \bar{H} as in Section 2. One can then easily check that the functor Γ transfers the set of cogenerators, and hence $\text{mod } \Sigma$ is residually small.

(\Leftarrow) Let $\text{mod } \Sigma$ be residually small. Then so is any $\text{mod } \Sigma' = \text{mod}(\Sigma, \text{Ens}')$. The functor \bar{H} transfers the set of cogenerators of $\text{mod}(\Sigma, \text{Ens}^{|C|})$ to a set of cogenerators in $\text{mod}(\Sigma, \hat{\mathbf{C}})$, and hence the latter is essentially bounded, by Lemma 3.7. Since monomorphisms in $\text{mod}(\Sigma, \mathbf{E})$ are also monomorphisms in $\text{mod}(\Sigma, \hat{\mathbf{C}})$ and the reflection functor preserves monomorphisms, essential monomorphisms in $\text{mod}(\Sigma, \mathbf{E})$ are also essential in $\text{mod}(\Sigma, \hat{\mathbf{C}})$, and hence $\text{mod}(\Sigma, \mathbf{E})$ is also essentially bounded. Then $\text{mod}(\Sigma, \mathbf{E})$ is residually small, by Lemma 3.8.

4. Injective algebras in $\text{mod}(\Sigma, \mathbf{E})$

4.1. Definition. In any category \mathbf{K} , *pushouts transfer monomorphisms* iff, for any pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ u \downarrow & & \downarrow v \\ C & \xrightarrow{g} & D \end{array}$$

whenever f is a monomorphism, then g is also a monomorphism. If \mathbf{K} has pushouts, one can easily check that this is equivalent to saying that, any diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ u \downarrow & & \\ C & & \end{array}$$

with f a monomorphism can be completed to a *commutative* diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ u \downarrow & & \downarrow v \\ C & \xrightarrow{g} & D \end{array}$$

with g a monomorphism. This is the condition (E4) in [1].

4.2. Proposition. *Pushouts transfer monomorphisms in $\text{mod}(\Sigma, \mathbf{E})$ iff they do in $\text{mod } \Sigma$.*

Proof. (\Rightarrow) Using the pair of adjoint functors $\Delta \dashv \Gamma$ given in the proof of Proposition 3.10 and the fact that Δ is faithful, one can easily check this.

(\Leftarrow) If the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ u \downarrow & & \downarrow v \\ C & \xrightarrow{g} & D \end{array}$$

is a pushout in $\text{mod}(\Sigma, \mathbf{E})$ with f a monomorphism, then, by the construction of pushouts in $\text{mod}(\Sigma, \mathbf{E})$, D is the reflection of some $P \in \text{mod}(\Sigma, \hat{\mathbf{C}})$ with

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 u \downarrow & & \downarrow v \\
 C & \xrightarrow{g} & P
 \end{array}$$

a pushout in $\text{mod}(\Sigma, \hat{C})$, $R\bar{g} = g$, and f a monomorphism. Now, for each $U \in C$,

$$\begin{array}{ccc}
 AU & \xrightarrow{f_U} & BU \\
 \downarrow & & \downarrow \\
 CU & \xrightarrow{g_U} & PU
 \end{array}$$

is a pushout in $\text{mod } \Sigma$ with f_U a monomorphism. Then, by the hypothesis on $\text{mod } \Sigma$, g_U (for all U), and hence \bar{g} are monomorphisms. Since $R\bar{g} = g$ and R preserves monomorphisms, g is a monomorphism.

4.3. Lemma. *The category $\text{mod}(\Sigma, E)$ has enough injectives iff, it is residually small and pushouts transfer monomorphisms.*

Proof. (\Rightarrow) To show that $\text{mod}(\Sigma, E)$ is residually small is to show it is essentially bounded, by Corollary 3.9. All the essential extensions of $A \in \text{mod}(\Sigma, E)$ can be embedded in any injective extension of A , hence there exists a set of essential extensions for A . To prove the second part, let $A \xrightarrow{f} B$ be a monomorphism and $A \xrightarrow{g} C$ any homomorphism. Embedding C into an injective E we get the following commutative diagram

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 g \downarrow & & \downarrow h \\
 C & \xrightarrow{j} & E
 \end{array}$$

where h is obtained because f is a monomorphism and E is injective. This, by an earlier remark, proves that pushouts transfer monomorphisms.

(\Leftarrow) For $A \in \text{mod}(\Sigma, E)$, take a maximal extension $f: A \rightarrow E$ of A , which exists by Lemma 3.2 (ii). We claim that E is injective. To prove this, let $g: B \rightarrow C$ be any monomorphism and $h: B \rightarrow E$ any homomorphism. Form the following pushout:

$$\begin{array}{ccc}
 B & \xrightarrow{g} & C \\
 h \downarrow & & \downarrow v \\
 E & \xrightarrow{u} & P
 \end{array}$$

By hypothesis u is a monomorphism, and hence retractable by Corollary 3.4. This proves that E is injective.

4.4. Proposition. *The category $\text{mod}(\Sigma, E)$ has enough injectives iff $\text{mod } \Sigma$ has enough injectives.*

Proof. This follows from Propositions 3.10 and 4.2 and Lemma 4.3.

This result substantially improves a similar result by Howlett [5]. Here, we deal with quasi-equational classes of algebras rather than equational classes as [5] does, but more importantly, our proof does not use the points of the topos whereas [5] only proves this result for a Grothendieck topos with enough points. Moreover, Proposition 3.10 provides a positive answer to Howlett's question [5, p. 108] whether essential boundedness of $\text{mod } \Sigma$ directly implies that of $\text{mod}(\Sigma, E)$.

5. Behaviour of injectivity in $\text{mod}(\Sigma, E)$

5.1. Banaschewski in [1] calls the notion of injectivity in a category \mathbf{K} *properly behaved* iff the following three propositions hold, which describe the relationship between essential boundedness, residual smallness and the existence of injective hulls in \mathbf{K} . Actually [1] deals with injectivity with respect to a more general type of morphism, but of course, here we only consider injectivity with respect to all monomorphisms.

(I) For any $A \in \mathbf{K}$ the following conditions are equivalent:

- (I1) A is injective.
- (I2) A is an absolute retract.
- (I3) A has no proper essential extensions.

(E) Every $A \in \mathbf{K}$ has an injective hull, unique up to isomorphisms.

(H) For any monomorphism $f: A \rightarrow B$, the following conditions are equivalent:

- (H1) $f: A \rightarrow B$ is an injective hull of A .
- (H2) $f: A \rightarrow B$ is a maximal essential extension.
- (H3) $f: A \rightarrow B$ is a minimal injective extension.

5.2. [1] also gives sufficient conditions for the proper behaviour of injectivity in \mathbf{K} as follows:

(E3) For any monomorphism $f: A \rightarrow B$, there exists a homomorphism $g: B \rightarrow C$ with gf an essential monomorphism.

(E4) Any diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \\ C & & \end{array}$$

with f a monomorphism can be completed to a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \downarrow v \\ C & \xrightarrow{u} & D \end{array}$$

such that u is a monomorphism.

(E5) Any direct limit of monomorphisms is a monomorphism.

(E6) The category \mathbf{K} is essentially bounded.

For $\mathbf{K} = \text{mod}(\Sigma, \mathbf{E})$, we now have the following counterpart of Proposition 5 in [1] for equational classes of algebras in Ens .

5.3. Proposition. *For $\text{mod}(\Sigma, \mathbf{E})$, the following are equivalent:*

- (i) *Injectivity is properly behaved.*
- (ii) *$\text{Mod}(\Sigma, \mathbf{E})$ has enough injectives.*
- (iii) *$\text{Mod}(\Sigma, \mathbf{E})$ is residually small and pushouts transfer monomorphisms.*
- (iv) *(E4) and (E6) are satisfied.*

Proof. (i) \Rightarrow (ii): By (E) in the definition of proper behaviour of injectivity.

(ii) \Rightarrow (iii): By Lemma 4.3.

(iii) \Rightarrow (iv): (E4) is trivial, by completing any diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \\ C & & \end{array}$$

to a pushout, and (E6) holds by Corollary 3.9.

(iv) \rightarrow (i): It remains to show (E3) and (E5). But Lemma 3.3 proves (E3), and (E5) is discussed in the proof of Lemma 3.2 (ii).

In particular, one has, by Proposition 4.4:

Injectivity is properly behaved in $\text{mod}(\Sigma, \mathbf{E})$ iff it is properly behaved in $\text{mod } \Sigma$.

We conclude with a couple of comments on injectivity in $\text{mod}(\Sigma, \mathbf{C})$. Since each of the functors

$$\text{mod}(\Sigma, \mathbf{E}) \xrightarrow{(V, -)} \text{mod } \Sigma \quad (V \in \mathbf{C})$$

has a left adjoint preserving monomorphisms, and by the well known fact that such a functor preserves injectives, if $A \in \text{mod}(\Sigma, \mathbf{E})$ is injective, then so is each AV ($V \in \mathbf{C}$). However, the converse of this is not true; for counter examples, in the case of abelian groups, the reader is referred to [2].

For certain Σ , one has characterizations of the injective $A \in \text{mod } \Sigma$ by properties of A in terms of its elements or its subobjects, for example: divisibility for abelian groups, completeness for Boolean algebras, and completeness and Booleanness for distributive lattices. An obvious question to ask is to what extent, that is for what \mathbf{E} , such characterizations remain valid in $\text{mod}(\Sigma, \mathbf{E})$. The only case where anything is known about this is that of abelian groups: divisibility = injectivity for abelian groups in the category $\text{Sh } \mathbf{L}$ of sheaves on a locale \mathbf{L} iff \mathbf{L} is Boolean [3].

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